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Differential properties of Meijer's G-function

E E Fitchard† and Victor Franco

Physics Department, Brooklyn College of the City University of New York, Brooklyn, New York 11210, USA

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Abstract. The kth derivative of any Meijer G-function whose argument is proportional to any rational power of the variable is obtained in terms of another G-function. Many known formulae for derivatives of G-functions are easily derived from our result. We also obtain, in passing, both an apparently new relationship for rearranging parameters of a special G-function, and the Fourier sine transform of a G-function whose argument is a rational power of the variable. Applications are made to several examples of physical and mathematical interest, such as the Holtsmark function used in stellar dynamics and plasma spectroscopy.

1. Introduction

Meijer's G-function (Meijer 1936, Erdelyi *et al* 1953, Mathai and Saxena 1973) has been given considerable attention in the mathematical literature‡, and has been employed recently in the study of physical problems (Milgram 1977, Thomas and Franco 1976, Franco 1979a,b). It has been useful in mathematical physics because of its analytical properties and because it can be expressed as a finite sum of generalised hypergeometric functions which have well-known series expansions. Consequently, the G-function is also relatively easy to compute numerically. Although many mathematical relationships involving the G-function are known, we have discovered a differential property which, as far as we know, is new and which is potentially useful in several branches of theoretical physics. Our result expresses the kth derivative of the G-function $G(\alpha z^r)$, where r is any rational number, as a single G-function. The expression we obtain is a generalisation, in two important respects, of known results involving the kth derivative of G-functions (Meijer 1941, 1952, Bhise 1962, Sundararajan 1966). First, no restriction is placed on the parameters of the G-function (other than those required by the definition, equation (1)). Second, the exponent r of the variable z is extended from an integer to any rational number. We surmise that all known derivative formulas for the G-function follow with relative ease from our results.

Special cases of the relationship derived here have previously proved useful in the study of physical problems, and new applications are not difficult to find. For example, in the Glauber approximation in atomic physics the scattering amplitude may be written in terms of kth derivatives of generating functions (Franco 1971, Thomas and Chan

† Present address: NL Petroleum Services DST, PO Box 60087, Houston, Texas 77205, USA.
‡ See Mathai and Saxena (1973) for example. In addition, most volumes of *Reviews of Mathematics* during the past decade contain reviews of papers on G-functions.

1973, Chan and Chang 1975, 1976, 1977, Kumar and Srivastava 1976), where k depends on the target atom and the transition under consideration. The generating functions, which depend on several variables, are written in different forms in different references. It has been shown (E E Fitchard, V Franco and B K Thomas, private communication) that each of these generating functions can be written in terms of Meijer's G-functions with arguments proportional to the square of the variables, and that a special case of one of the results derived here can be used to eliminate the kth derivative in each case.

Another application, not considered previously, is the Holtsmark function used in stellar dynamics (Chandrasekhar and von Neumann 1942, 1943) and plasma spectroscopy (Hey and Griem 1975) which we evaluate in closed form. In this case derivatives of a G-function with an argument proportional to the inverse sixth power of the variable are required. This new application is discussed in § 4, where we also obtain the Fourier sine transform of the G-function in closed form.

The main results, equations (6), (9), (11a), (11b) and (14), are presented in § 2. This is followed by a demonstration, in § 3, that many known derivative formulae for the G-function are, or follow easily from, special cases of the present results. The paper is concluded in § 4 with a few representative applications.

2. A new differential property of the G-function

In this section we derive a result for the kth derivative of a G-function whose argument is proportional to an integral power of the independent variable. We then show how the result can be extended to any rational power of the variable.

Meijer's G-function is defined by a Mellin–Barnes-type contour integral as (Erdelyi *et al* 1953)

$$G^{m,n}_{p,q}\left(z\middle|{}^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right)\equiv\frac{1}{2\pi\mathrm{i}}\int_L\frac{\Pi_{j=1}^{m}\Gamma(b_j-s)\,\Pi_{j=1}^{n}\Gamma(1-a_j+s)}{\Pi_{j=m+1}^{q}\Gamma(1-b_j+s)\,\Pi_{j=n+1}^{p}\Gamma(a_j-s)}z^s\,\mathrm{d}s \tag{1a}$$

$$\equiv\frac{1}{2\pi\mathrm{i}}\int_L\chi(s)z^s\,\mathrm{d}s, \tag{1b}$$

where an empty product is interpreted as 1, $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$, and the parameters are such that none of the poles of $\Gamma(b_j-s)$, $j=1,\ldots,m$, coincides with the poles of $\Gamma(1-a_j+s)$, $j=1,\ldots,n$. Three different contours L in the complex plane are discussed in Erdelyi *et al* (1953). The essential features of each are that the contour must separate the ascending chains of poles due to $\Gamma(b_j-s)$, $j=1,\ldots,m$, from the descending chains of poles due to $\Gamma(1-a_j+s)$, $j=1,\ldots,n$, and if more than one of these contours are applicable, they are equivalent. The variable z and the parameters a_j, $j=1,\ldots,p$, and b_j, $j=1,\ldots,q$, may be complex. Equations (1a) and (1b) define $\chi(s)$. To simplify notation the G-function is often written

$$G^{mn}_{pq}\left(z\middle|{}^{a_p}_{b_q}\right),\ G^{mn}_{pq}(z),\quad \text{or } G(z).$$

To determine the kth derivative with respect to z of a G-function with argument αz^r, where α is a complex constant and r is a positive integer, we employ equation (1b) and interchange the order of differentiation and integration. Thus

$$z^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}G^{mn}_{pq}\left(\alpha z^r\middle|{}^{a_p}_{b_q}\right)=\frac{1}{2\pi\mathrm{i}}\int_L\chi(s)\alpha^s z^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}z^{rs}\,\mathrm{d}s. \tag{2}$$

Using

$$z^k \frac{\mathrm{d}^k}{\mathrm{d}z^k} z^{rs} = \frac{\Gamma(1+rs)}{\Gamma(1-k+rs)} z^{rs} \tag{3}$$

and the multiplication formula for the Γ-function

$$\Gamma(rz) = (2\pi)^{1/2(1-r)} r^{rz-1/2} \prod_{j=0}^{r-1} \Gamma\left(z + \frac{j}{r}\right), \qquad r = 1, 2, \ldots, \tag{4}$$

we obtain

$$z^k \frac{\mathrm{d}^k}{\mathrm{d}z^k} G_{pq}^{mn}(\alpha z^r|_{b_q}^{a_p}) = \frac{r^k}{2\pi \mathrm{i}} \int_L \chi(s) \alpha^s \frac{\Pi_{j=0}^{r-1} \Gamma(s+(1+j)/r)}{\Pi_{j=0}^{r-1} \Gamma(s+(1+j-k)/r)} z^{rs}\, \mathrm{d}s. \tag{5}$$

The restriction that none of the poles due to $\Gamma(s+(1+j)/r)$, $j = 0, \ldots, r-1$, coincides with the poles of $\Gamma(b_j - s)$, $j = 1, \ldots, m$, is not required here, since all the poles of $\Pi_{j=0}^{r-1} \Gamma(s+(1+j)/r)$ coincide with poles of $\Pi_{j=0}^{r-1} \Gamma(s+(1+j-k)/r)$ and are, therefore, removable singularities. Equation (5) can now be written

$$z^k \frac{\mathrm{d}^k}{\mathrm{d}z^k} G_{pq}^{mn}(\alpha z^r|_{b_q}^{a_p}) = r^k G_{p+r,q+r}^{m,n+r}(\alpha z^r|_{b_1,\ldots,b_q,\Delta(r,k)}^{\Delta(r,0),a_1,\ldots,a_p}), \tag{6}$$

where the symbol $\Delta(r, c)$ is defined by

$$\Delta(r, c) \equiv \{c/r, (c+1)/r, \ldots, (c+r-1)/r\}, \tag{7}$$

and thus represents a set of r parameters. $\Delta(0, c)$ is defined to be the empty set. Equation (6) is the first new differential property of the G-function.

To obtain the second differential relationship we first note an elementary property of the G-function (Erdelyi *et al* 1953, p 209):

$$G_{pq}^{mn}(z^{-1}|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}) = G_{qp}^{nm}(z|_{1-a_1,\ldots,1-a_p}^{1-b_1,\ldots,1-b_q}). \tag{8}$$

If we use this property in equation (6), and note that

$$\{1-c/r, 1-(c+1)/r, \ldots, 1-(c+r-1)/r\} = \Delta(r, 1-c),$$

we obtain the second differential relationship for the G-function, namely

$$z^k \frac{\mathrm{d}^k}{\mathrm{d}z^k} G_{pq}^{mn}\left(\frac{\alpha}{z^r}\middle|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right) = r^k G_{p+r,q+r}^{m+r,n}\left(\frac{\alpha}{z^r}\middle|_{\Delta(r,1),b_1,\ldots,b_q}^{a_1,\ldots,a_p,\Delta(r,1-k)}\right). \tag{9}$$

Thus equations (6) and (9) give the result of the kth derivative of $G(\alpha z^r)$ when r is any positive or negative integer.

Equations (6) and (9) can be further generalised with the aid of Mathai and Saxena (1973, equation (1.2.5)†):

$$G_{pq}^{mn}(z|_{b_q}^{a_p}) = (2\pi)^{(1-l)c^*} l^U G_{lp,lq}^{lm,ln}(z^l l^{l(p-q)}|_{\Delta(l,b_1),\ldots,\Delta(l,b_q)}^{\Delta(l,a_1),\ldots,\Delta(l,a_p)}), \tag{10a}$$

where

$$c^* = m + n - \tfrac{1}{2}p - \tfrac{1}{2}q, \qquad U = \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i + \tfrac{1}{2}p - \tfrac{1}{2}q + 1 \tag{10b}$$

† There is a misprint in the definition of U in this equation. It should be as given in our equation (10b).

and l is a positive integer. This result is easily obtained by means of equations (1) and (4) and $\Gamma(z)\equiv\Gamma(rz/r)$. If we replace z by $\alpha z^{r/l}$, where r is any integer, we have

$$G_{pq}^{mn}(\alpha z^{r/l}|_{b_q}^{a_p})=(2\pi)^{(1-l)c^*}l^U G_{lp,lq}^{lm,ln}(\alpha^l l^{l(p-q)} z^r|_{\Delta(l,b_1),\ldots,\Delta(l,b_q)}^{\Delta(l,a_1),\ldots,\Delta(l,a_p)}). \tag{10c}$$

This implies that a G-function whose argument is proportional to a rational power of z may be expressed in terms of a G-function whose argument is proportional to an integral power of z. Combining equations (6) and (10c) we obtain

$$z^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}G_{pq}^{mn}(\alpha z^{r/l}|_{b_q}^{a_p})$$
$$=(2\pi)^{(1-l)c^*}l^U r^k G_{lp+r,lq+r}^{lm,ln+r}(\alpha^l l^{l(p-q)} z^r|_{\Delta(l,b_1),\ldots,\Delta(l,b_q),\Delta(r,k)}^{\Delta(r,0),\Delta(l,a_1),\ldots,\Delta(l,a_p)}) \tag{11a}$$

for any positive integers r and l. Similarly, by combining equations (9) and (10c) we obtain

$$z^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}G_{pq}^{mn}\left(\frac{\alpha}{z^{r/l}}\middle|_{b_q}^{a_p}\right)=(2\pi)^{(1-l)c^*}l^U r^k G_{lp+r,lq+r}^{lm+r,ln}\left(\frac{\alpha^l l^{l(p-q)}}{z^r}\middle|_{\Delta(r,1),\Delta(l,b_1),\ldots,\Delta(l,b_q)}^{\Delta(l,a_1),\ldots,\Delta(l,a_p),\Delta(r,1-k)}\right) \tag{11b}$$

for any positive integers r and l. Thus equations (11a) and (11b) give the results for the kth derivative of a G-function whose argument is proportional to *any rational power* of z.

Equations (6), (9), (11a) and (11b) may be expressed in alternative forms. For example, if we note an elementary property of the G-function (Mathai and Saxena 1973, equation (1.2.2)†),

$$G_{p+1,q+1}^{m+1,n}(z|_{1,b_1,\ldots,b_q}^{a_1,\ldots,a_p,1-t})=(-1)^t G_{p+1,q+1}^{m,n+1}(z|_{b_1,\ldots,b_q,1}^{1-t,a_1,\ldots,a_p}),\qquad t=0,1,2,\ldots, \tag{12a}$$

and set $t=1$, we have

$$G_{p+1,q+1}^{m+1,n}(z|_{1,b_1,\ldots,b_q}^{a_1,\ldots,a_p,0})=-G_{p+1,q+1}^{m,n+1}(z|_{b_1,\ldots,b_q,1}^{0,a_1,\ldots,a_p}). \tag{12b}$$

This result may be generalised by considering the following G-function with its Mellin–Barnes integral representation:

$$G_{p+r,q+r}^{m,n+r}(z|_{b_1,\ldots,b_q,\Delta(r,k)}^{\Delta(r,0),a_1,\ldots,a_p})$$
$$=\frac{1}{2\pi\mathrm{i}}\int_L\frac{\Pi_{j=1}^m\Gamma(b_j-s)\,\Pi_{j=1}^n\Gamma(1-a_j+s)\,\Pi_{j=1}^r\Gamma(1+s-(j-1)/r)}{\Pi_{j=m+1}^q\Gamma(1-b_j+s)\,\Pi_{j=n+1}^p\Gamma(a_j-s)\,\Pi_{j=1}^r\Gamma(1+s-(k+j-1)/r)}z^s\,\mathrm{d}s, \tag{13a}$$

where r and k are positive integers. The products of Γ-functions involving r in equation (13a) may be written as

$$\frac{\Pi_{j=1}^r\Gamma(1+s-(j-1)/r)}{\Pi_{j=1}^r\Gamma(1+s-(k+j-1)/r)}=\prod_{j=0}^{r-1}\frac{\Gamma(s+(1+j)/r)}{\Gamma(s+(1+j-k)/r)}$$
$$=r^{-k}\Gamma(1+rs)/\Gamma(1+rs-k)$$
$$=(-r)^{-k}(-rs)_k$$
$$=(-1)^k\prod_{j=0}^{r-1}\frac{\Gamma(-s+(k+j)/r)}{\Gamma(-s+j/r)},$$

† There is a misprint in this equation. The first lower parameter on the left-hand side should be unity instead of zero.

where Pochhammer's symbol $(a)_k$ is defined as

$$(a)_k \equiv a(a+1)(a+2)\dots(a+k-1), \qquad (a)_0 \equiv 1.$$

Therefore we obtain, as a generalisation of equation (12b), the result

$$G^{m,n+r}_{p+r,q+r}\left(z\,\middle|\,{}^{\Delta(r,0),a_1,\dots,a_p}_{b_1,\dots,b_q,\Delta(r,k)}\right) = (-1)^k G^{m+r,n}_{p+r,q+r}\left(z\,\middle|\,{}^{a_1,\dots,a_p,\Delta(r,0)}_{\Delta(r,k),b_1,\dots,b_q}\right). \tag{13b}$$

(We believe this is a new result for rearranging the parameters of the G-function of equation (13a).) If we apply equations (8) and (13b) to equation (9), we obtain an alternative form for equation (9), namely

$$z^k \frac{\mathrm{d}^k}{\mathrm{d}z^k} G^{mn}_{pq}\left(\frac{\alpha}{z^r}\,\middle|\,{}^{a_p}_{b_q}\right) = (-r)^k G^{m,n+r}_{p+r,q+r}\left(\frac{\alpha}{z^r}\,\middle|\,{}^{\Delta(r,1-k),a_p}_{b_q,\Delta(r,1)}\right). \tag{14}$$

Corresponding alternative forms for equations (6), (11a) and (11b) may be similarly obtained.

In the next section we show that many of the known results for derivatives of G-functions are easily obtained from equations (6) and (14).

3. Special cases

Setting α and r equal to unity in equations (6) and (14) immediately yields two results (Meijer 1952, equations (24) and (25)).

If $b_1 = 0$ and $\alpha = r = 1$, equation (6) yields

$$z^k \frac{\mathrm{d}^k}{\mathrm{d}z^k} G^{mn}_{pq}\left(z\,\middle|\,{}^{a_1,\dots,a_p}_{0,b_2,\dots,b_q}\right) = G^{m,n+1}_{p+1,q+1}\left(z\,\middle|\,{}^{0,a_1,\dots,a_p}_{0,b_2,\dots,b_q,k}\right) \tag{15a}$$

$$= (-1)^k G^{m+1,n}_{p+1,q+1}\left(z\,\middle|\,{}^{a_1,\dots,a_p,0}_{k,0,b_2,\dots,b_q}\right) \tag{15b}$$

$$= (-1)^k G^{mn}_{pq}\left(z\,\middle|\,{}^{a_1,\dots,a_p}_{k,b_2,\dots,b_q}\right), \qquad m \geqslant 1. \tag{15c}$$

Equation (15b) follows from equation (13b) in the special case $r = 1$, and equation (15c) follows from equation (15b) after use is made of the G-function property (Erdelyi *et al* 1953, p 209)

$$G^{mn}_{pq}\left(z\,\middle|\,{}^{a_1,\dots,a_{p-1},\alpha}_{\alpha,b_2,\dots,b_q}\right) = G^{m-1,n}_{p-1,q-1}\left(z\,\middle|\,{}^{a_1,\dots,a_{p-1}}_{b_2,\dots,b_q}\right), \qquad m, p, q \geqslant 1, \tag{16}$$

together with the property that the G-function of equation (1a) is symmetric in the parameters $b_1, \dots, b_m$. If in equation (15c) we replace a_i by $a_i - b_1$ and b_j by $b_j - b_1$ for $i = 1, \dots, p$ and $j = 2, \dots, q$, and use the property (Erdelyi *et al* 1953)

$$z^\sigma G^{mn}_{pq}\left(z\,\middle|\,{}^{a_p}_{b_q}\right) = G^{mn}_{pq}\left(z\,\middle|\,{}^{a_p+\sigma}_{b_q+\sigma}\right) \tag{17}$$

with $\sigma = -b_1$, and then replace z by βz, we obtain

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(z^{-b_1} G^{mn}_{pq}\left(\beta z\,\middle|\,{}^{a_p}_{b_q}\right)\right) = (-1)^k z^{-b_1-k} G^{mn}_{pq}\left(\beta z\,\middle|\,{}^{a_1,\dots,a_p}_{b_1+k,b_2,\dots,b_q}\right), \qquad m \geqslant 1, \tag{18}$$

due to Meijer (1941) (see also Bhise 1962). Three other results of the same type, also due to Meijer (1941) (see also Bhise 1962), are obtained similarly after setting $a_1 = \alpha = r = 1$ in equation (14), $b_q = 0$ and $\alpha = r = 1$ in equation (6), and $a_p = \alpha = r = 1$ in equation (14).

As a final special case, let $a_1, \ldots, a_r (r \leq n)$ be in arithmetic progression with difference $-1/r$. Then from equations (17), (14) and (16) and the properties of the symbol $\Delta(r, c)$ defined in equation (7), we obtain the result

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(z^{r(a_1-1)}G_{pq}^{mn}\left(\frac{\alpha}{z^r}\middle|\begin{matrix}a_1, a_1-1/r, \ldots, a_1-(r-1)/r, a_{r+1}, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right)\right)$$

$$= \alpha^{a_1-1} z^{-k}(-r)^k G_{pq}^{mn}\left(\frac{\alpha}{z^r}\middle|\begin{matrix}\Delta(r, 1-k), a_{r+1}-a_1+1, \ldots, a_p-a_1+1\\ b_1-a_1+1, \ldots, b_q-a_1+1\end{matrix}\right) \tag{19a}$$

$$= (-r)^k z^{r(a_1-1)-k} G_{pq}^{mn}\left(\frac{\alpha}{z^r}\middle|\begin{matrix}\Delta(r, 1-k+ra_1-r), a_{r+1}, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) \tag{19b}$$

$$= (-r)^k z^{r(a_1-1)-k} G_{pq}^{mn}\left(\frac{\alpha}{z^r}\middle|\begin{matrix}a_1-k/r, a_2-k/r, \ldots, a_r-k/r, a_{r+1}, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) \tag{19c}$$

for $r \leq n$. (The sequence $a_{r+1}, \ldots, a_p$ is empty if $r = p$.) This expression and three others, which are of the same type and which can be obtained in a similar manner, are due to Sundararajan (1966).

4. Applications

Applications of the results in § 2 are legion, as will be demonstrated in this section by considering several examples of physical and mathematical interest.

Consider first the kth derivative of $z^\beta/(1+az^r)^\alpha$, where r is a positive integer and a, α and β are complex. By means of equation (17) and the relationship (Mathai and Saxena 1973, p 54)

$$(1-z)^{-b} = G_{11}^{11}(-z|_{0}^{1-b})/\Gamma(b), \tag{20a}$$

we may write

$$\frac{z^\beta}{(1+az^r)^\alpha} = a^{-\beta/r} G_{11}^{11}(az^r|_{\beta/r}^{1-\alpha+\beta/r})/\Gamma(\alpha). \tag{20b}$$

Combining equations (6) and (20b) yields

$$z^k \frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z^\beta}{(1+az^r)^\alpha}\right) = \frac{a^{-\beta/r}}{\Gamma(\alpha)} r^k G_{1+r,1+r}^{1,1+r}(az^r|_{\beta/r, \Delta(r,k)}^{\Delta(r,0), 1-\alpha+\beta/r}). \tag{21}$$

As an example of the usefulness of this procedure, we note that a particularly simple case of the left-hand side of equation (21) was recently encountered in an analytic evaluation of an important integral in collision theory (Detrich and Conn 1977). The technique used was a repeated integration by parts. Similar applications involving the more general expression given by the left-hand side of equation (21) may be done with the aid of that equation. Furthermore, this procedure can be generalised to include the k-fold integration by parts of any function which can be written in the form $G(az^r)\, \mathrm{d}^k f(z)/\mathrm{d}z^k$. The kth derivative of the G-function needed for the integration by parts is obtained from equation (6) as a single G-function. The integral of a function $f(z)$ times a G-function is given in the literature (Erdelyi *et al* 1953, Mathai and Saxena 1973) for many of the important functions used in mathematical physics. If r is a negative integer in $z^\beta/(1+az^r)^\alpha$, an expression similar to equation (21) is obtained with the aid of equation (14). Furthermore, if use is made of equations (11*a*) and (11*b*), r can be extended to any rational number. Thus we see that the expressions presented here in conjunction with the method of integration by parts can be used to evaluate some interesting integrals.

In the Glauber approximation in atomic physics, the scattering amplitude for charged particle–atom collisions is the product of derivatives, with respect to a set of parameters λ_i, of a generating function. The number of parameters and the order of the derivatives depend on the target atom and the transition under consideration. The generating function either is or can be (Franco 1971, Thomas and Chan 1973) written as a sum of products of G-functions whose arguments are proportional to the squares of the parameters λ_i. The differentiation here can be carried out explicitly by applying equation (6) in the special case $r=2$.

In the study of stellar dynamics (Chandrasekhar and von Neumann 1942, 1943) and plasma spectroscopy (Hey and Griem 1975) the Holtsmark function $H(\beta)$ and its derivatives are of interest. This function, defined by (Chandrasekhar and von Neumann 1942, 1943, Hey and Griem 1975)

$$H(\beta)\equiv\frac{2}{\pi\beta}\int_0^\infty \mathrm{d}x\,\exp\left[-\left(\frac{x}{\beta}\right)^{3/2}\right]x\sin x,$$

can be expressed in terms of a G-function by making the substitution (Mathai and Saxena 1973, p 53)

$$\exp[-(x/\beta)^{3/2}]=G_{01}^{10}((x/\beta)^{3/2}|0),$$

so that

$$H(\beta)=\frac{2}{\pi}\int_0^\infty \mathrm{d}x(\sin x)G_{01}^{10}\left(\left(\frac{x}{\beta}\right)^{3/2}\bigg|\frac{2}{3}\right). \tag{22}$$

A more general integral, which can also be used to evaluate functions related (Hey and Griem 1975) to the Holtsmark function, will be considered here. Define the Fourier sine transform of the G-function of $\alpha x^{2k/\rho}$,

$$I\equiv\int_0^\infty \mathrm{d}x(\sin\gamma x)G_{pq}^{mn}(\alpha x^{2k/\rho}|_{b_q}^{a_p}), \tag{23}$$

where γ is real, k and ρ are positive integers, and α is complex. Applying equation (10*c*) to this G-function yields

$$I=(2\pi)^{(1-\rho)c^*}\rho^U\int_0^\infty \mathrm{d}x(\sin\gamma x)G_{\rho p,\rho q}^{\rho m,\rho n}(\alpha^\rho\rho^{\rho(p-q)}x^{2k}|_{\Delta(\rho,b_q)}^{\Delta(\rho,a_p)}), \tag{24}$$

where c^* and U are defined by equation (10*b*), and $\Delta(\rho, a_p)$ is a shorthand notation for $\Delta(\rho, a_1), \Delta(\rho, a_2), \ldots, \Delta(\rho, a_p)$, and similarly for $\Delta(\rho, b_q)$.

To evaluate the integral I, we use equation (1*b*) for the G-function in equation (24), and we exchange the order of integrations. This leads to the Mellin transform (Erdelyi *et al* 1954)

$$\int_0^\infty \mathrm{d}x\,x^{2ks}\sin\gamma x=\gamma^{-2ks-1}\Gamma(1+2ks)\sin[\pi(1+2ks)/2],$$

$$\gamma>0,\quad -1/k<\mathrm{Re}(s)<0. \tag{25}$$

The case $\gamma<0$ is obtained trivially from equations (23) and (25). The restriction on $\mathrm{Re}(s)$ leads to restrictions on some of the parameters of the G-function in equation (23), which will be discussed below. From equation (25) and the well-known relationship $\Gamma(z)\Gamma(1-z)=\pi/\sin \pi z$, equation (24) becomes

$$I=(2\pi)^{(1-\rho)c^*}\rho^{U}\frac{1}{2\mathrm{i}\gamma}\int_L \mathrm{d}s\,\chi(s)\left(\frac{\alpha^{\rho}\rho^{\rho(p-q)}}{\gamma^{2k}}\right)^s\frac{\Gamma(1+2ks)}{\Gamma((1+2ks)/2)\Gamma((1-2ks)/2)}. \tag{26}$$

By using the duplication formula for $\Gamma(2z)$ and then using the multiplication formula equation (4), it follows that

$$\frac{\Gamma(1+2ks)}{\Gamma((1+2ks)/2)\Gamma((1-2ks)/2)}=\frac{2^{2ks}}{\pi^{1/2}}\frac{\Gamma(1+ks)}{\Gamma(\frac{1}{2}-ks)} \tag{27a}$$

$$=\frac{(2k)^{2ks+1/2}}{(2\pi)^{1/2}}\prod_{j=0}^{k-1}\frac{\Gamma(s+(1+j)/k)}{\Gamma(-s+(\frac{1}{2}+j)/k)}. \tag{27b}$$

On substituting equation (27b) into equation (26), using the definition of the G-function equation (1a), and the definition equation (7), we obtain the result

$$\int_0^\infty \mathrm{d}x(\sin\gamma x)G^{mn}_{pq}\left(\alpha x^{2k/\rho}\middle|^{a_p}_{b_q}\right)$$

$$=\frac{(k\pi)^{1/2}}{\gamma}(2\pi)^{(1-\rho)c^*}\rho^{U}G^{\rho m,\rho n+k}_{\rho p+2k,\rho q}\left(\left(\frac{2k}{\gamma}\right)^{2k}\alpha^{\rho}\rho^{\rho(p-q)}\middle|^{\Delta(k,0),\Delta(\rho,a_p),\Delta(k,\frac{1}{2})}_{\Delta(\rho,b_q)}\right),$$

$$p+q<2(m+n),\quad |\arg\alpha|<\pi c^*,\quad \gamma>0,\quad k>0,\quad \rho>0. \tag{28}$$

The restriction on $\mathrm{Re}(s)$ in equation (25) leads to the restrictions $\mathrm{Re}(a_j)<1$, $j=1,\ldots,n$ and $\mathrm{Re}(b_j)>-\rho/k$, $j=1,\ldots,m$. Equation (28) gives a closed-form result for the Fourier sine transform of $G(\alpha x^{2k/\rho})$ for positive rational exponents $2k/\rho$. Although it is not unlikely that this result has appeared explicitly elsewhere, we are not aware that it has. The integral I can also be calculated using the result for the Hankel transform of $G(\alpha x^{2k/\rho})$ given in Mathai and Saxena (1973), after expressing $\sin\gamma x$ in terms of $J_{1/2}(\gamma x)$.

By setting $\gamma=m=q=1$, $n=p=0$, $\alpha=\beta^{-3/2}$, $k=3$, $\rho=4$ and $b_1=\frac{2}{3}$ in equation (28), we obtain from equations (22) and (28) the result for the Holtsmark function,

$$H(\beta)=\frac{2^{7/3}}{\pi^2}\left(\frac{3}{2}\right)^{1/2}G^{43}_{64}\left(\frac{1}{4}\left(\frac{3}{\beta}\right)^6\middle|^{\Delta(3,0),\Delta(3,\frac{1}{2})}_{\Delta(4,\frac{2}{3})}\right). \tag{29}$$

Thus we have shown that the Holtsmark function is a Meijer G-function.

To evaluate derivatives of the Holtsmark function, we apply equation (14) to equation (29), and we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\beta^n}H(\beta)=\frac{2^{7/3}}{\pi^2}\left(\frac{3}{2}\right)^{1/2}\left(-\frac{6}{\beta}\right)^n G^{4,9}_{12,10}\left(\frac{1}{4}\left(\frac{3}{\beta}\right)^6\middle|^{\Delta(6,1-n),\Delta(3,0),\Delta(3,\frac{1}{2})}_{\Delta(4,\frac{2}{3}),\Delta(6,1)}\right). \tag{30a}$$

Since elements of $\Delta(6,1-n)$ and $\Delta(3,0)$ differ by an integer, this result is not convenient for numerical evaluation. It can be written in a more convenient form by using equations (7), (8) and (13b) to obtain the result

$$\frac{\mathrm{d}^n}{\mathrm{d}\beta^n}H(\beta)=\frac{2^{7/3}}{\pi^2}\left(\frac{3}{2}\right)^{1/2}\left(\frac{6}{\beta}\right)^n G^{3,10}_{10,12}\left(4\left(\frac{\beta}{3}\right)^6\middle|^{\Delta(6,0),\Delta(4,\frac{1}{3})}_{\Delta(3,1),\Delta(3,\frac{1}{2}),\Delta(6,n)}\right). \tag{30b}$$

From equation (16) this reduces to

$$\frac{d^n}{d\beta^n}H(\beta)=\frac{2^{7/3}}{\pi^2}\left(\frac{3}{2}\right)^{1/2}\left(\frac{6}{\beta}\right)^n G^{3,7}_{7,9}\left(4\left(\frac{\beta}{3}\right)^6\middle|\begin{matrix}\Delta(3,0),\Delta(4,\frac{1}{3})\\ \Delta(3,1),\Delta(6,n)\end{matrix}\right). \tag{30c}$$

Since no two elements of the set $\Delta(3,1)$ differ by an integer, the G-function in equation (30c) can be written (Erdelyi *et al* 1953) as the sum of three ${}_7F_8$ hypergeometric functions. This form facilitates numerical evaluation, since the generalised hypergeometric function ${}_pF_{p+1}(z)$ converges quite rapidly. Note that, for $n=0,1,2,3,4$, equation (1b) can be used to further reduce equation (30c).

As another example we note that the Airy function, which is used widely in optics (Nussenzveig 1969), surface physics (Sahni *et al* 1977) and semi-classical rainbow scattering (Ford and Wheeler 1959), for example, can be expressed as a G-function

$$\mathrm{Ai}(z^r)=G^{20}_{02}(z^{3r}/9|0,\tfrac{1}{3})/2\pi 3^{1/6}. \tag{31}$$

From this and equation (6), we obtain for the kth derivative of $\mathrm{Ai}(z^r)$ the result

$$z^k\frac{d^k}{dz^k}\mathrm{Ai}(z^r)=\frac{(3r)^k}{2\pi 3^{1/6}}G^{2,3r}_{3r,3r+2}\left(\frac{z^{3r}}{9}\middle|\begin{matrix}\Delta(3r,0)\\ 0,\frac{1}{3},\Delta(3r,k)\end{matrix}\right), \tag{32}$$

where r is a positive integer.

As a final application we note that certain products of Bessel functions can be written as a single G-function (Mathai and Saxena 1973),

$$z^\mu I_\nu(z)J_\nu(z)=\pi^{1/2}2^{3\mu/2}G^{10}_{04}(z^4/64|\mu/4+\nu/2,\mu/4-\nu/2,\mu/4,\mu/4+\tfrac{1}{2}), \tag{33}$$

from which we obtain the result

$$z^k\frac{d^k}{dz^k}(z^\mu I_\nu(z)J_\nu(z))=\pi^{1/2}2^{3\mu/2}4^k G^{14}_{48}\left(\frac{z^4}{64}\middle|\begin{matrix}\Delta(4,0)\\ \mu/4+\nu/2,\mu/4-\nu/2,\mu/4,\mu/4+\frac{1}{2},\Delta(4,k)\end{matrix}\right). \tag{34}$$

In most of these applications only the case where r is an integer has been considered, but it is not difficult to extend them to the case where r is a rational number by using equations (10c) and (14).

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